The problems SAT and HAMILTON are equivalent under linear parsimonious reductions in the plane

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Abstract

It is conjectured that there exists no linear reduction from the Hamiltonian Cycle problem to the Satisfiability problem. In contrast to the general case, we show that the planar Hamiltonian cycle problem is linearly and parsimoniously reducible to the planar Satisfiability problem. Since the converse is already known to be true, the two problems are equivalent under linear parsimonious reductions.

Key words: Computational Complexity, Combinatorial Problems, Planarity, Linear Time, Parsimonious Reductions.

1 Introduction

Many NP-complete problems, (e.g., SAT, 3-COL, KERNEL, 3DM) are not only equivalent under polynomial reductions, but also under linear reductions, i.e. reductions computable in *linear time* on Random Access Machine [3–6].[1] Moreover, the transformations can often be made parsimonious, i.e. such that they realize a one-to-one correspondence between the solutions of the instances. In this note, as in [1,2], we investigate *exact (i.e. linear and parsimonious) reductions.* The interest of such a notion is that if two problems A and B are *exactly* equivalent, i.e. *exactly* reducible each other, then A and B have exactly the same complexity according to: (a) the number of

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[1] In particular, [6] argues the robustness of linear time complexity on RAMS. Other authors [11,8] have studied non-deterministic quasi-linear time (NQL), involving reductions computable in time $O(n \log^{O(1)} n)$ on Turing Machines.

their solutions, and (b) the enumeration of their solutions (delays between two consecutive solutions are the same up to a multiplicative constant).

Given a connected graph G, *the HAM problem* asks whether a (so called Hamiltonian) cycle can visit each vertex of G exactly once. There are many variants of this problem: the graph G may be directed or undirected, we may want a path instead of a cycle, and its ends may be specified or not. If G is planar, this induces as many variants for the PLAN-HAM problem. In [1], we showed that all the cited HAM (resp. PLAN-HAM) variants are *exactly* equivalent.

Given a formula F of the propositional logic in conjunctive normal form, the *SAT problem* asks whether F has a satisfying assignment. A SAT instance can be seen as a bipartite graph problem, in which there is one vertex v_x per variable x, one vertex v_c per clause c, and one edge (v_x, v_c) per occurrence of x or $\neg x$ in c. If this graph is planar, then we have a PLAN-SAT instance.

Theorem 1 *PLAN-SAT and PLAN-HAM are* exactly *equivalent, i.e.:*
(a) PLAN-SAT is exactly *reducible to PLAN-HAM.*
(b) PLAN-HAM is exactly *reducible to PLAN-SAT.*

We showed (a) in [1], and we show (b) in this note. The result is surprising because SAT is believed to be too weak [9] to linearly express the hamiltonicity property in the general case (which contrasts with its NQL-completeness [11]).

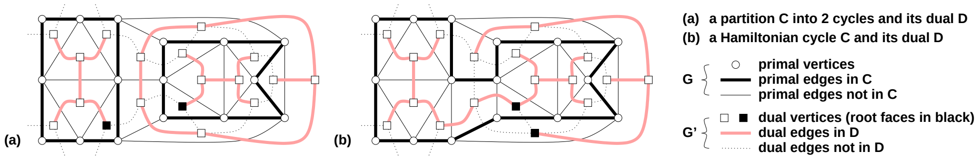

Fig. 1. A planar graph G and its dual graph G'

Here is an intuition (see Figure 1): Firstly, given a connected planar graph G, our SAT system builds a partition $\mathcal{C}$ of G into k disjoint cycles by constraining each vertex to have exactly two incident edges in $\mathcal{C}$. The second idea is to force $k = 1$ (i.e. that $\mathcal{C}$ is a Hamiltonian cycle) with the help of the dual graph G'. $\mathcal{C}$ induces a structure $\mathcal{D}$ in G', which is the graph of faces that are left connected by $\mathcal{C}$. Lemma 14 shows that $\mathcal{C}$ is a Hamiltonian cycle iff $\mathcal{D}$ is a forest of exactly two trees. So, our SAT system further constrains each face but two to elect exactly one father among its adjacent faces in the same connected component of $\mathcal{D}$. Both faces without father are the roots of the only two trees of $\mathcal{D}$, but $\mathcal{D}$ may still contain additional connected components that are cycles possibly adorned with entering trees. These cycles do not elect roots, and hence they cannot be detected by the SAT system. Fortunately, Lemma 16 shows that any cycle in $\mathcal{D}$ prevents the two roots to be adjacent. So, by constraining the roots to be adjacent, we indirectly forbid cycles in $\mathcal{D}$, and we get a linear reduction PLAN-HAM $\leq$ SAT, which is even parsimonious, thanks to a trick to fix the roots *a priori*. The SAT system is then easily made planar.

2 A local characterization of planar Hamiltonicity

Fact 2 (generalized Jordan curve theorem) *Let $\mathcal{C}$ be a collection of k disjoint simple closed curves $C_1, \cdots, C_k$ lying in a sphere S. Then $S \setminus \mathcal{C}$ falls into exactly $k+1$ maximal connected regions (or regions, for short).*

We are now given a connected planar graph $G(V, E, F)$ embedded in a sphere S, with V (resp. E, F) as its set of vertices (resp. edges, faces).

Definition 3 (k-Ham partition, Ham-cycle) *A k-Ham partition $\mathcal{C}$ of G is a collection of k disjoint simple cycles $C_1(V_1, E_1), \cdots, C_k(V_k, E_k)$ in G, such that $(V_1, \cdots, V_k)$ is a partition of V. If $k = 1$, then $\mathcal{C}$ is a Ham-cycle of G. For some fixed k-Ham partition $\mathcal{C}$, an edge $e \in E$ is said to be thick (resp. thin) in G if $\mathrm{e} \in \cup E_i$ (resp. $\mathrm{e} \notin \cup E_i$), $1 \leq i \leq k$.*

Definition 4 (dual graph) *For each edge $e \in E$ with adjacent faces f_1 and f_2, we define $e' = dual(e)$ to be a new associated edge (f_1, f_2), where f_1 and f_2 are seen as vertices. We also note $e = primal(e')$. The dual graph $G'(V', E', F')$ of G, denoted $dual(G)$, is defined as follows: $V' = F$, $F' = V$, and $E' = \{dual(e), e \in E\}$. Each vertex $v' \in V'$ is embedded in the sphere S such that v' lies in the region R induced by its associated face $f \in F$, and each edge $e' = dual(e) \in E'$ is embedded in S such that e' crosses e exactly once and does not cross G elsewhere.*

Definition 5 (Ham-dual) *Let $G' = dual(G)$ and $\mathcal{C}$ be a k-Ham partition of G. The Ham-dual $\mathcal{D}$ of $\mathcal{C}$ is defined to be G' after removal of all edges $e' = dual(e)$ such that e is thick in G, i.e. belongs to $\mathcal{C}$. The edges in $\mathcal{D}$ (resp. in $G' \setminus \mathcal{D}$) are called the thick (resp. thin) edges of G', in other words, $e' = dual(e)$ is thick (thin) iff e is thin (thick). Maximal connected components in $\mathcal{D}$ are called components, for short.*

From now on, we are given a k-Ham partition $\mathcal{C}$ in G, along with its Ham-dual $\mathcal{D}$ in $G' = dual(G)$.

Claim 6 *$\mathcal{D}$ has exactly $k' = k+1$ components.*

PROOF. It immediately follows from Fact 2. $\square$

Definition 7 (C-regions) *Let C be a simple cycle in G or G'. From Fact 2, $S \setminus C$ falls into two regions R_1 and R_2, called the two C-regions, such that R_1, R_2, and C form a partition of the sphere S.*

Definition 8 (C-chords, C-edges, R-edges) *Let C be a simple cycle in G or G', R be an arbitrary C-region, and $e = (x, y) \in E$ be an edge lying in*

$C \cup R$ such that $x \in C$. If $y \in R$, then e is called an R-edge. Otherwise, $y \in C$ and e is called a C-edge (resp. a C-chord) if $e \in C$ (resp. if $e \notin C$). Note that, if C belongs to $\mathcal{C}$, then all its C-edges are thick, and since the cycles in $\mathcal{C}$ are disjoint, all its R-edges and C-chords are thin.

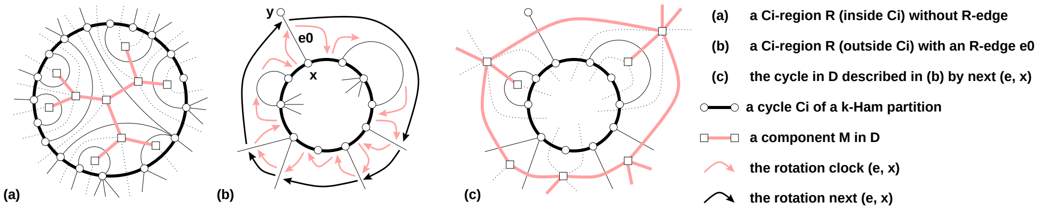

Fig. 2. Acyclic and cyclic components in $\mathcal{D}$

Claim 9 *Let C_i be a cycle in $\mathcal{C}$, and R be an arbitrary C_i-region, such that there does not exist any R-edge. Then the unique component $M \in \mathcal{D}$ lying in R is acyclic, i.e. is a tree (see Figure 2a).*

PROOF. Since no R-edge lies in R and G is planar and connected, the set $V_R \subset V$ of vertices lying in R is empty. Thus, R may only contains C_i-chords. Now, suppose that M is cyclic: let C' be a cycle in M, e' be an arbitrary edge of C' and $e = (x, y) = primal(e')$. Notice that x and y lie in the two distinct C'-regions. Since exactly one of the two C'-regions does not contain C_i, either x or y must lie in R, which contradicts $V_R = \emptyset$. $\square$

Claim 10 *Let C_i be a cycle in $\mathcal{C}$, and R be an arbitrary C_i-region, such that there exists an R-edge $e_0 \in E$. Then, the component M of $\mathcal{D}$ containing $e'_0 = dual(e_0)$ is cyclic (see Figures 2(b,c)).*

PROOF. For any edge $e = (x, y)$, $clock(e, x)$ denotes the edge after e in the clockwise order around x. Also, $other(e, x)$ denotes y, and vice-versa. Let $I(e_0)$ be the iteration `e:=e`$_0$`(x,y); loop { e:=clock(e,x); write(e); x:=other(e,x); }`. If we stop $I(e_0)$ as soon as e_0 is met again, then $I(e_0)$ writes the boundary of a face f in the anti-clockwise order. This face f, denoted $left(e_0, x)$, is the left-hand face for an observer located at y and looking towards x. Now, suppose that e_0 is an R-edge of C_i with $x \in \mathcal{C}$. Then, the first R-edge e met after e_0 in $I(e_0)$ is denoted by $e = next(e_0, x)$. Therefore, e_0 and $next(e_0, x)$ share the same adjacent face $f = left(e_0, x)$.

If C_i is a cycle of $\mathcal{C}$, and e is an R-edge of C_i, then $common(e, C_i)$ denotes the unique common vertex of e and C. Let $J(e_0, C_i)$ be the iteration `e:=e`$_0$`(x,y); loop { write(left(e,x),dual(e)); e:=next(e,x); x:=common(e,C`$_i$`); } until(e=e`$_0$`)`. The loop terminates because of the invariant $x \in C_i$. Since each encountered edge e shares a common face $left(e, x)$ with $next(e, x)$, $J(e_0, C_i)$ writes the successive vertices and edges of a cycle C' in G' (possibly non-simple if some articulation point x is met). Since all edges of C' are the dual edges of R-edges, and since R-edges are always thin in G, it follows that the edges of C' are all thick in G'. This means that the component M lying in R and containing $e'_0 = dual(e_0)$ also contains the cycle C'. $\square$

From now on, an arbitrary outer-face $f_{out} \in F$ is chosen in G.

Definition 11 (interior, exterior) *Let C be a simple cycle in G, and R_1, R_2 be the two C-regions. Exactly one of them, say R_2, contains f_{out}. Then R_1 (resp. R_2) is said to be the interior (resp. exterior) of C, and is denoted $int(C)$ (resp. $ext(C)$).*

Definition 12 (nested cycles) *For any two cycles $C_i, C_j \in \mathcal{C}$ such that $int(C_i) \subset int(C_j)$, we note $C_i < C_j$ (which is read: C_i is nested in C_j, or C_j nests C_i). Moreover, if there is no $C_h \in \mathcal{C}$ such that $C_i < C_h < C_j$, then we note $C_j = succ(C_i)$ and $brothers(C_i) = \{C_j \in \mathcal{C}, succ(C_j) = succ(C_i)\}$. Finally, a min-cycle (resp. max-cycle) C_i is a cycle of $\mathcal{C}$ which nests (resp. is nested in) no other cycle of $\mathcal{C}$.*

Lemma 13 *Let M be a component in $\mathcal{D}$. Then M is a tree iff: (1) M lies in the interior of a min-cycle C_i of $\mathcal{C}$, or (2) M lies in the exterior of a cycle C_j which is the unique max-cycle of $\mathcal{C}$.*

PROOF. There are exactly four cases:

- If M lies in $R = int(C_i)$ and C_i is a min-cycle of $\mathcal{C}$, then Claim 9 applies to C_i and R, and M is a tree.
- If M lies in $R = ext(C_j)$ and C_j is the unique max-cycle of $\mathcal{C}$, then Claim 9 applies to C_j and R, and M is a tree.
- If M lies in $R = ext(C_i)$ and C_i is a non-unique max-cycle of $\mathcal{C}$, then since G is planar and connected, there exists $e = (x, y) \in E$ such that $x \in C_i$ and $y \in C_j$ for some max-cycle $C_j \neq C_i$. Since e is an R-edge, Claim 10 applies to C_i and R, and M is cyclic.
- Otherwise, M lies in $ext(C_i) \cap int(C_j)$, for some $C_j = succ(C_i)$. Since G is planar and connected, there exists $e = (x, y) \in E$ such that $x \in C_i$ and $y \in C_j$ or $y \in C_h$ where $C_h \in brothers(C_i)$, $h \neq i$. In both cases, e is an R-edge for $R = ext(C_i)$, so Claim 10 applies to C_i and R, and M is cyclic. $\square$

Lemma 14 *$k = 1$, i.e. $\mathcal{C}$ is a Ham-Cycle iff $\mathcal{D}$ consists of exactly two trees.*

PROOF. $(\Longrightarrow)$: If $k = 1$, then let C_1 be the unique cycle in $\mathcal{C}$: by Claim 6, $\mathcal{D}$ has $k' = k + 1 = 2$ components M_1 and M_2, lying in $int(C_1)$ and $ext(C_1)$

respectively. Since C_1 is both a min-cycle and the unique max-cycle in $\mathcal{C}$, Lemma 13 applies twice, and M_1 and M_2 are both trees.

$(\Longleftarrow)$: If $k' = 2$, then by Claim 6, $k = k' - 1 = 1$. $\square$

Definition 15 (twin components) *Two components M_1 and M_2 in $\mathcal{D}$ are said to be twin iff the two following conditions hold: (1) both M_1 and M_2 are trees, and (2) there are two vertices $r_1 \in M_1$ and $r_2 \in M_2$, called the twin roots of M_1 and M_2, such that r_1 and r_2 are neighbor, i.e. $(r_1, r_2) \in E'$.*

Lemma 16 *$k = 1$, i.e. $\mathcal{C}$ is a Ham-cycle iff two twin components exist in $\mathcal{D}$.*

PROOF. $(\Longrightarrow)$: It is an immediate consequence of Lemma 14. We only have to exhibit two twin roots r_1 and r_2: Choose an arbitrary edge e in the unique cycle $C_1 \in \mathcal{C}$, and let $e' = (f, g) = dual(e)$, such that f lies in $int(C_1)$ and g lies in $ext(C_1)$, and set $r_1 = f$ and $r_2 = g$.

$(\Longleftarrow)$: Assume M_1 and M_2 are two twin components with respective twin roots r_1 and r_2. Since both components are trees, then by Lemma 13, each one must lie either in the interior of a min-cycle or in the exterior of the unique max-cycle. Suppose $k > 1$, then there are two cases:

- $M_1 \in int(C_i)$ and $M_2 \in int(C_j)$ where C_i and C_j are disjoint min-cycles in $\mathcal{C}$. Because G is planar and C_i and C_j are disjoint, any path from r_1 to r_2 in G' must contain an intermediate vertex lying in $ext(C_i) \cap ext(C_j)$. So, r_1 and r_2 are not twin roots, a contradiction.
- $M_1 \in int(C_i)$ and $M_2 \in ext(C_j)$ where C_i is a min-cycle and C_j is the unique max-cycle in $\mathcal{C}$. Since $k > 1$, we have $C_i \neq C_j$. Because G is planar and C_i and C_j are disjoint, any path from r_1 to r_2 in G' contains a third vertex lying in $ext(C_i) \cap int(C_j)$. So, r_1 and r_2 are not twin roots. $\square$

3 The reduction PLAN-HAM $\leq$ PLAN-SAT

To get a parsimonious reduction, we need to fix the twin roots r_1 and r_2 inside two predetermined faces of G. From the proof of Lemma 16, r_1 and r_2 may be arbitrarily chosen among any two neighbor faces separated by a thick edge. In order to fix such a thick edge, we choose an arbitrary vertex $v \in V$ of any degree, and we explode it into the subgraph gadget $G(v)$ of Figure 3b. A one-to-one correspondence suggested by Figures 3(c,d) holds between the Hamiltonian cycles in G before and after the substitution, and G now contains a vertex t of degree 2 which forces its two incident edges to be thick in any Hamiltonian cycle. Thus, r_1 and r_2 can be fixed as the two adjacent faces of t.

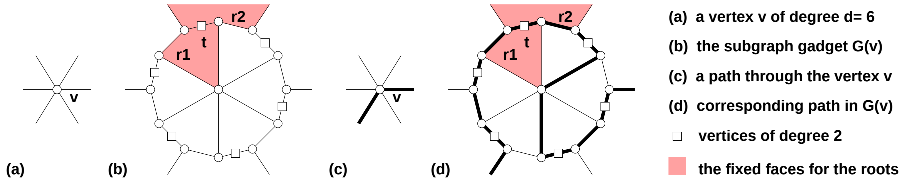

Fig. 3. Explosion of a vertex of degree $d = 6$

For the sake of readability, we assume that the special clauses $1/N(\ell_1, \cdots, \ell_d)$ and $2/N(\ell_1, \cdots, \ell_d)$ are SAT subsystems that are satisfied iff exactly one (resp. two) literal among ℓ_i ($1 \leq i \leq d$), are assigned true. Implementing them linearly and parsimoniously with planar clauses is an easy task detailed in Appendix A. The reduction is as follows:

- *Set of variables:* Each edge $e \in E \cup E'$ has an associated boolean variable $thick_e$, which is true iff e is thick in $\mathcal{C} \cup \mathcal{D}$. The choice of r_1 and r_2 determines a direction for the forest $\mathcal{D}$, so each vertex $f \in V'$ of degree d has d associated boolean variables $father_f^{e'}$, one for each edge $e' = (f, g) \in E'$, which is true iff $e' \in \mathcal{D}$ and g is the father of f in $\mathcal{D}$.
- *$\mathcal{C}$ is a k-Ham partition of G:* For each vertex $v \in V$ of degree d, with incident edges $e_1, \cdots, e_d$, generate the constraint $2/N(thick_{e_1}, \cdots, thick_{e_d})$.
- *$\mathcal{D}$ is the Ham-dual of $\mathcal{C}$:* For each edge $e \in E$ and $e' = dual(e)$, generate the clauses $(thick_e \vee thick_{e'})$ and $(\neg thick_e \vee \neg thick_{e'})$.
- *Each face other than r_1 and r_2 has exactly one father:* For each vertex $f \in V'$ of degree d, $f \notin \{r_1, r_2\}$, with incident edges $e'_1, \cdots, e'_d$, generate the constraint $1/N(father_f^{e'_1}, \cdots, father_f^{e'_d})$.
- *Both twin roots r_1 and r_2 have no father:* For each edge $e' \in E'$ incident to a root $r \in \{r_1, r_2\}$, generate unit clause $(\neg father_r^{e'})$.
- *The directed Ham-dual $\mathcal{D}$ is consistent:* For each edge $e' = (f, g) \in E'$, generate the constraints $(father_f^{e'} \implies thick_{e'})$, $(father_g^{e'} \implies thick_{e'})$, $(thick_{e'} \implies father_f^{e'} \vee father_g^{e'})$, and $(\neg father_g^{e'} \vee \neg father_f^{e'})$.

The system is linear in $|G+G'|$ and its correctness is an immediate consequence of Lemmas 14 and 16. Moreover, Figure 4 shows how to embed the SAT system in each face so that it can be made planar, by using a linear number of parsimonious PLAN-SAT crossover-boxes (this standard mechanism [7,10] embeds one variable in each corner of a square, and forces the opposite corner variables to have the same assignment). Thus, Theorem 1 is proved.

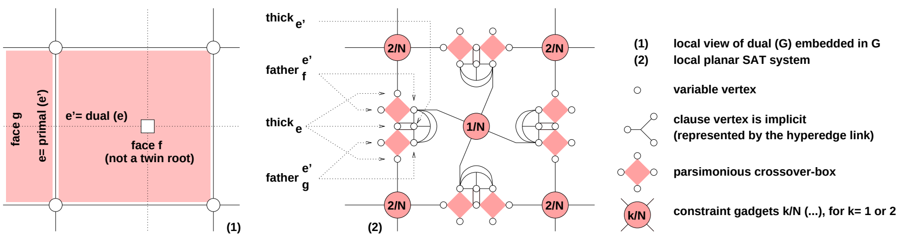

Fig. 4. Local planar embedding of the SAT system in a face

4 Conclusion

Linear reductions allow to investigate the strong relationships between the complexity of hard (NP-complete) problems, and hence to classify them more precisely than polynomial or even quasi-linear reductions. Further, *exact* reductions allow to show that two combinatorial problems have the same complexity w.r.t the structure of their set of solutions. Surprisingly, we have shown the *exact* equivalence of PLAN-HAM and PLAN-SAT. We now hope to enlarge this *exact* class to many other combinatorial planar problems [2].

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APPENDIX

A Planar encoding of the constraints 1/N and 2/N

We encode the constraint $1/N(x_1, \cdots, x_d)$ by simulating the pair of finite boolean sequences $([a_k], [x_k])$ where $[a_k]$ is a monotone non-increasing boolean sequence starting with one, and where $x_k = 1$ iff k is the last rank such that $a_k = 1$, e.g. $[a_k] = [1, 1, 1, 0, 0, 0]$ and $[x_k] = [0, 0, 1, 0, 0, 0]$. The monotonicity of $[a_k]$ is encoded by the unit clause (a_1) and the clauses $(a_{k+1} \implies a_k)$, for any $k < d$. The sequence $[x_k]$ is then encoded by $(x_d \iff a_d)$ and $(x_k \iff a_k \wedge \neg a_{k+1})$ for any $k < d$, which develop into the clauses $(\neg x_d \vee a_d)$, $(\neg a_d \vee x_d)$, $(\neg x_k \vee a_k)$, $(\neg x_k \vee \neg a_{k+1})$, $(\neg a_k \vee a_{k+1} \vee x_k)$. A planar embedding of $1/N(\cdots)$ is the chain of tetrahedrons shown in Figure A.1a.

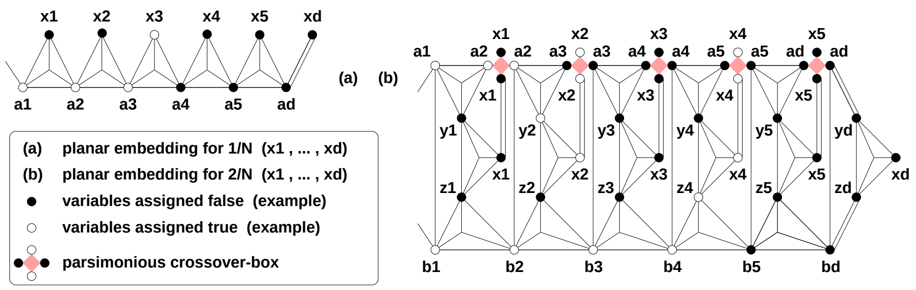

Fig. A.1. Planar embedding of constraints $1/N$ and $2/N(x_1, \cdots, x_d)$, for $d = 6$

The encoding of the constraint $2/N(x_1, \cdots, x_d)$ follows the same idea. We simulate two pairs of finite boolean sequences $([a_k], [y_k])$ and $([b_k], [z_k])$ the same way as above, so that the sequences $[y_k]$ and $[z_k]$ contain exactly one "one". Then, we define $[x_k]$ by $(x_k \iff y_k \vee z_k)$ for any $k \leq d$, which develops into $(\neg x_k \vee y_k \vee z_k)$, $(\neg y_k \vee x_k)$, $(\neg z_k \vee x_k)$. Moreover, we want $y_i = 1$ and $z_j = 1$ for $i \neq j$, so we constrain the initial subsequence of ones in $[a_k]$ to be strictly longer than the initial subsequence of ones in $[b_k]$ by adding the clauses $(a_k \vee \neg b_k)$ and $(\neg y_k \vee \neg z_k)$ for any $k \leq d$. A possible planar embedding of $2/N(\cdots)$ is the triple chain of tetrahedrons shown in Figure A.1b, where d parsimonious crossover-boxes have been added in order to embed a copy of $[x_k]$ on the boundary of the outer face.